ON BETHE VECTORS IN THE sl_{N+1} GAUDIN MODEL

S. CHMUTOV AND I. SCHERBAK

Abstract. The note deals with the Gaudin model associated with the tensor product of n irreducible finite-dimensional sl_{N+1}-modules marked by distinct complex numbers $z_1, \ldots, z_n$. The Bethe Ansatz is a method to construct common eigenvectors of the Gaudin hamiltonians by means of chosen singular vectors in the factors and z_j's. These vectors are called Bethe vectors.

The question if the Bethe vectors are non-zero vectors is open. By the moment, the only way to verify that was based on a relation to critical points of the master function of the Gaudin model, and non-triviality of a Bethe vector was proved only in the case when the corresponding critical point is non-degenerate ([ScV, MV1]). However degenerate critical points do appear in the Gaudin model ([ReV, Section12]).

We believe that the Bethe vectors never vanish, and suggest an approach that does not depend on non-degeneracy of the corresponding critical point. The idea is for a Bethe vector to choose a suitable subspace in the weight space and to check that the projection of the Bethe vector to this subspace is non-zero. We apply this approach to verify non-triviality of Bethe vectors in new examples.

1. Introduction

We study the Gaudin model of statistical mechanics associated with the Lie algebra $sl_{N+1}(\mathbb{C})$. The *space of states* of the model is the tensor product

$$L = L_{\Lambda(1)} \otimes \cdots \otimes L_{\Lambda(n)} ,$$

where $L_{\Lambda(j)}$ is a finite-dimensional irreducible sl_{N+1}-module with highest weight $\Lambda(j)$, $1 \le j \le n$. For the standard notions of representation theory see [FuH].

In the Gaudin model, the modules $L_{\Lambda(1)}, \ldots, L_{\Lambda(n)}$ are the spin spaces of n particles located at distinct points $z_1, \ldots, z_n \in \mathbb{C}$. Write $z = (z_1, \ldots, z_n)$. The *Gaudin hamiltonians* $H_1(z), \ldots, H_n(z)$ are mutually commuting linear operators in L which are defined as follows,

$$H_j(z) = \sum_{i \neq j} \frac{C_{ij}}{z_j - z_i} , \quad 1 \le j \le n ,$$

here C_{ij} acts as the Casimir operator on factors $L_{\Lambda(i)}$ and $L_{\Lambda(j)}$ of the tensor product and as the identity on all other factors.

One of the main problems in the Gaudin model is simultaneous diagonalization of the operators $H_1(z), \ldots, H_n(z)$. The Gaudin hamiltonians commute with the diagonal action of sl_{N+1} in L, therefore it is enough to find common eigenvectors and the eigenvalues in the subspace of singular vectors of a given weight, for every weight.

The algebraic Bethe Ansatz is a method to construct such vectors. The idea is to find some function $\mathbf{v} = \mathbf{v}(\mathbf{t})$ taking values in the weight subspace ($\mathbf{t}$ is a multidimensional auxiliary variable) and to determine a certain special value of its argument, $\mathbf{t}^{(0)}$, in such a way that $\mathbf{v}(\mathbf{t}^{(0)})$ is a common eigenvector of the hamiltonians. The equations on $\mathbf{t}$ which determine these special values of the argument are called *the Bethe equations*, and $\mathbf{v}(\mathbf{t}^{(0)})$ is called *the Bethe vector*. For the Gaudin model, the Bethe equations and the function $\mathbf{v}(\mathbf{t})$ are written in [FeFRe, ReV, SV]. On Bethe vectors in the Gaudin model see also [G, FaT, Re].

It was believed that for generic z one can find an eigenbasis in the subspace of singular vectors consisting of Bethe vectors only. This is indeed the case for the tensor products of $sl_2(\mathbb{C})$-modules and for the tensor products of several copies of first and last fundamental sl_{N+1}-modules ([ScV, MV1]). Recent results of [MV2] show however that generically other eigenvectors have to be present in eigenbases as well. These other vectors are in some sense "more degenerate" than Bethe vectors, see [F1] and especially [F2, Section 5.5] discussing the "degeneracies".

In the Bethe Ansatz, two problems naturally arise: to find solutions of the Bethe equations, and to check non-triviality of the corresponding Bethe vectors. Both problems are open and seem to be difficult ones. On solutions to the Bethe equations in some particular cases, see [V, ScV, MV1, Sc].

The present note is devoted to the question if Bethe vectors are non-zero vectors. By the moment, the only known way to verify that was extremely non-direct, via the so-called *master function*. Namely, it appeared that the Bethe equations in the Gaudin model form the critical point system of a certain function $S(\mathbf{t}; z)$, here $\mathbf{t}$ is a multidimensional variable and $z = (z_1, \ldots, z_n)$ is fixed, [ReV]. Moreover, the norm of the Bethe vector $\mathbf{v}(\mathbf{t}^{(0)})$ with respect to some (degenerate) bilinear form on the tensor product turned out to be the Hessian of $S(\mathbf{t}; z)$ at the critical point $\mathbf{t}^{(0)}$; hence the Bethe vectors corresponding to non-degenerate critical points of the function $S(\mathbf{t}; z)$ appeared to be non-zero vectors, [V, MV1]. In this way, the non-triviality of Bethe vectors has been checked for generic z in the case of tensor products of $sl_2(\mathbb{C})$-modules and in the case of tensor products of several copies of first and last fundamental sl_{N+1}-modules, [ScV, MV1].

It is known however, that for some values of z the master function does have degenerate critical points; an example can be found in [ReV, Section 12]. Notice that in that example the corresponding Bethe vector is a non-zero vector as well. We believe that the Bethe vectors are always non-trivial.

Conjecture. *In the $sl_{N+1}(\mathbb{C})$ Gaudin model, every Bethe vector is non-zero, for any z. For some values of z the number of Bethe vectors (i.e. of solutions to the Bethe equations, i.e. of critical points of the master function) may decrease, but the Bethe vectors still are non-zero.*

We suggest a more direct approach that does not depend on non-degeneracy of the corresponding critical point. The idea is to project a Bethe vector to a suitable subspace in the space of singular vectors of a given weight and to check that the projection is non-zero.

We exploit this idea in some examples of tensor products of irreducible finite-dimensional sl_{N+1}-modules. The case of the tensor product of $n = 2$ modules is special. First of all, in this case *all* values of z are generic. Indeed, as it was pointed out in [ReV, Section 5], for any fixed $z_1 \neq z_2$ the linear change of variables $\mathbf{u} = (\mathbf{t} - z_1)/(z_2 - z_1)$ turns the Bethe system on $\mathbf{t}$ with $z = (z_1, z_2)$ into the Bethe system on $\mathbf{u}$ with $z = (0, 1)$. Next, the Gaudin hamiltonias $H_1(0,1) = -H_2(0,1)$ are reduced to the Casimir operator, and hence act in any irreducible submodule of the tensor product by multiplication by a constant, i.e. *any* singular vector is their common eigenvector. Finally, non-triviality of a Bethe vector for $n > 2$ in many cases can be deduced from non-triviality of a certain set of Bethe vectors corresponding to $n = 2$ and $z = (0, 1)$, by means of *iterated singular vectors* introduced in [ReV]; see [Sc] for a more detailed explanation.

Let $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$ be the tensor product of two irreducible finite-dimensional sl_{N+1}-modules, where $L_{\Lambda(1)}$ is marked by $z_1 = 1$ and $L_{\Lambda(0)}$ by $z_0 = 0$. Denote simple positive roots of sl_{N+1} by $\alpha_1, \dots, \alpha_N$.

In our first example (Section 4.1), we consider arbitrary integral dominant weights $\Lambda(1)$, $\Lambda(0)$ and assume $\mathbf{v}_k$ to be a Bethe vector of the weight $\Lambda(1) + \Lambda(0) - k\alpha_1$.

In another example (Section 4.3), we restrict $L_{\Lambda(0)}$ to be a symmetric power of the standard sl_{N+1}-representation, and assume $\mathbf{v}_{k,1,1}$ to be a Bethe vector in L of the weight $\Lambda(1) + \Lambda(0) - k\alpha_1 - \alpha_2 - \alpha_3$.

Theorem. *Bethe vectors $\mathbf{v}_k$ and $\mathbf{v}_{k,1,1}$ are non-trivial.*

For $N = 1$, any Bethe vector is of the form $\mathbf{v}_k$, therefore the example 4.1 implies that for $N = 1$ and $n = 2$ the Bethe vectors never vanish. Moreover, this example admits an immediate generalization to $n > 2$, see Theorem 3 in Section 4.2. As a corollary we obtain that if $L_{\Lambda(0)} = m\lambda_1$ and $\Lambda(k_1, k_2) = \Lambda(1) + \Lambda(0) - k_1\alpha_1 - k_2\alpha_2$ is the highest weight of an irreducible component of L, then Bethe vectors of the weight $\Lambda(k_1, k_2)$ do not vanish. In particular, if $L_{\Lambda(1)}$ and $L_{\Lambda(0)}$ are sl_3-representations, then all Bethe vectors in L do not vanish.

In the examples 4.1, 4.3 the subspace of singular vectors is one-dimensional, therefore a Bethe vector, if it exists, gives an eigenbasis of the Gaudin hamiltonians in the corresponding weight subspace. A way to solve the Bethe equations in the example 4.3 is explained in [Sc].

The key ingredient of our proof is to write the Bethe equations and projections of vector $\mathbf{v}(\mathbf{t})$ in terms of symmetric functions in $\mathbf{t}$, see Section 3.3 and Section 4. Our calculations are based on funny relations between symmetric rational functions which generalize the "Jacobi identity"

$$\frac{1}{(x-y)(x-z)} + \frac{1}{(y-x)(y-z)} = \frac{-1}{(z-x)(z-y)},$$

see Theorems 1 and Corollary 1 in Section 2.

Plan of the note. Section 2 is devoted to the "Jacobi-like" identities; Section 3 contains a description of the Bethe equations and Bethe vectors; in Section 4 we verify non-triviality of Bethe vectors in the examples.

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2. Identities

For a function $g(t_1 \dots t_k)$, define *symmetrization* as follows,

$$\mathcal{S}ym_k[g] := \sum_{\pi\in S_k} g\big(\pi(t_1),\dots,\pi(t_k)\big)\,,$$

here the sum runs over the group S_k of all permutations π of variables $t_1 \dots t_k$.

Theorem 1. *For any fixed s_1, s_2 and s, we have*

$$\text{(I)}\qquad \mathcal{S}ym_k\left[\frac{1}{(s_1-t_1)(t_1-t_2)\dots(t_{k-1}-t_k)(t_k-s_2)}\right]=$$

$$=\frac{(-1)^k\cdot(s_1-s_2)^{k-1}}{(s_1-t_1)\dots(s_1-t_k)\cdot(s_2-t_1)\dots(s_2-t_k)}\ ,$$

$$\text{(II)}\qquad \mathcal{S}ym_k\left[\frac{1}{(s-t_1)(t_1-t_2)\dots(t_{k-1}-t_k)t_k}\right]=\frac{s^{k-1}}{(s-t_1)\dots(s-t_k)\cdot t_1\dots t_k}\ ,$$

$$\text{(III)}\qquad \mathcal{S}ym_k\left[\frac{1}{(t_1-t_2)(t_2-t_3)\dots(t_{k-1}-t_k)(t_k-s)}\right]=\frac{(-1)^k}{(s-t_1)\dots(s-t_k)}\ .$$

Proof. We prove the first and the third identities by induction in k. The second identity can be obtained from the first one by substitution $s_1 = s$ and $s_2 = 0$.

The first identity for $k = 1$ becomes

$$\mathcal{S}ym_1\left[\frac{1}{(s_1-t_1)\cdot(t_1-s_2)}\right]=\frac{1}{(s_1-t_1)\cdot(t_1-s_2)}=\frac{-1}{(s_1-t_1)\cdot(s_2-t_1)}\,,$$

and is true. Suppose that the identity (I) holds for $k-1$ and prove it for k. Consider the subgroup $S_{k-1} \subset S_k$ of permutations acting on the first $k-1$ variables. Every summand in the symmetrization of our fraction has a form

$$\frac{1}{(s_1-t_{i_1})(t_{i_1}-t_{i_2})\dots(t_{i_{k-1}}-t_{i_k})(t_{i_k}-s_2)}\ .$$

Combine together all the summands with a fixed value of i_k, say $i_k = j$, and factor out the last multiplier $1/(t_j - s_2)$. Then we can write

$$\mathcal{S}ym_k \left[\frac{1}{(s_1-t_1)(t_1-t_2)\dots(t_{k-1}-t_k)(t_k-s_2)} \right] =$$

$$= \sum_{j=1}^{k} \mathcal{S}ym_{k-1} \left[\frac{1}{(s_1-t_{i_1})(t_{i_1}-t_{i_2})\dots(t_{i_{k-1}}-t_j)} \right] \cdot \frac{1}{t_j-s_2} ,$$

here the values of $i_1, \dots, i_{k-1}$ are different form j and the group S_{k-1} acts by the permutations which keep t_j. By the induction hypothesis this is

$$\sum_{j=1}^{k} \frac{(-1)^{k-1} \cdot (s_1-t_j)^{k-2}}{(s_1-t_1)\dots\widehat{(s_1-t_j)}\dots(s_1-t_k) \cdot (t_j-t_1)\dots\widehat{(t_j-t_j)}\dots(t_j-t_k)} \cdot \frac{1}{t_j-s_2} ,$$

where the "hat" means that the corresponding factor is omitted. Multiplying this expression with $(s_1-t_1)\dots(s_1-t_k)\cdot(s_2-t_1)\dots(s_2-t_k)$ we get

$$(-1)^k \sum_{j=1}^{k} \left((s_1-t_j)^{k-1} \cdot \frac{(s_2-t_1)\dots\widehat{(s_2-t_j)}\dots(s_2-t_k)}{(t_j-t_1)\dots\widehat{(t_j-t_j)}\dots(t_j-t_k)} \right) .$$

This is nothing but the Lagrange interpolation formula for a polynomial of degree $k-1$ in a variable s_2 that takes the value $(-1)^k(s_1-t_j)^{k-1}$ at the point $s_2 = t_j$ for every $j = 1, \dots, k$. Therefore it is equal to $(-1)^k(s_1-s_2)^{k-1}$.

The third identity is obvious for $k = 1$,

$$\mathcal{S}ym_1 \left[\frac{1}{(t_1-s)} \right] = \frac{1}{(t_1-s)} = \frac{-1}{(s-t_1)} .$$

Suppose that the identity (II) holds for $k-1$ and prove it for k. As before, combining together all the summands of the left hand side with a fixed variable t_j at the last factor of the denominator we get

$$\mathcal{S}ym_k \left[\frac{1}{(t_1-t_2)\dots(t_{k-1}-t_k)(t_k-s)} \right] =$$

$$= \sum_{j=1}^{k} \mathcal{S}ym_{k-1} \left[\frac{1}{(s_1-t_{i_1})(t_{i_1}-t_{i_2})\dots(t_{i_{k-1}}-t_j)} \right] \cdot \frac{1}{t_j-s} ,$$

where the values of $i_1, \dots, i_{k-1}$ are different form j and the group S_{k-1} acts by the permutations that keep t_j. By the induction hypothesis this is equal to

$$\sum_{j=1}^{k} \frac{(-1)^{k-1}}{(t_j-t_{i_1})(t_j-t_{i_2})\dots(t_j-t_{i_{k-1}})} \cdot \frac{1}{t_j-s} .$$

Multiplying this expression with $(s-t_1)\dots(s-t_k)$ we get

$$(-1)^k \sum_{j=1}^{k} \frac{(s-t_{i_1})(s-t_{i_2})\dots(s-t_{i_{k-1}})}{(t_j-t_{i_1})(t_j-t_{i_2})\dots(t_j-t_{i_{k-1}})}\ ,$$

where the indices $i_1,\dots,i_{k-1}$ in every summand are the integers between 1 and k different form j. Recognizing in the last expression the Lagrange interpolation formula we conclude that this is exactly $(-1)^k$. □

It is convenient to write identities on functions which are symmetric with respect to variables $t_1,\dots,t_k$ in terms of the elementary symmetry functions.

Notation.

$$T(x) = (x-t_1)\ \dots\ (x-t_k) = x^k - \tau_1 x^{k-1} + \ \dots\ + (-1)^k \tau_k\,,$$

that is τ_i is the i-th elementary symmetric function in $t_1,\dots,t_k$ for $1 \le i \le k$; we set $\tau_0 = 1$.

With this notation, the identities of Theorem 1 take the form

$$(\mathrm{I}') \qquad \mathcal{S}ym_k\left[\frac{1}{(s_1-t_1)(t_1-t_2)\dots(t_{k-1}-t_k)(t_k-s_2)}\right] = \frac{(-1)^k\cdot(s_1-s_2)^{k-1}}{T(s_1)\cdot T(s_2)}\ ,$$

$$(\mathrm{II}') \qquad \mathcal{S}ym_k\left[\frac{1}{(s-t_1)(t_1-t_2)\dots(t_{k-1}-t_k)t_k}\right] = \frac{s^{k-1}}{T(s)\ \tau_k}\ ,$$

$$(\mathrm{III}') \qquad \mathcal{S}ym_k\left[\frac{1}{(t_1-t_2)(t_2-t_3)\dots(t_{k-1}-t_k)(t_k-s)}\right] = \frac{(-1)^k}{T(s)}\ .$$

Corollary 1. *We have*

$$(\mathrm{IV}) \qquad \mathcal{S}ym_k\left[\frac{1}{(t_1-t_2)\dots(t_{i-2}-t_{i-1})(t_{i-1}-s)(s-t_i)(t_i-t_{i+1})\dots(t_{k-1}-t_k)t_k}\right] =$$

$$= \frac{(-1)^{i-1}\ s^{k-i}\ \tau_{i-1}}{T(s)\ \tau_k}\ , \qquad \textit{for} \quad 1 \le i \le k\,.$$

Proof. The formula (IV) for $i=1$ is exactly the identity (II$'$). For $2 \le i \le k$, first let us take the sum over the subgroup $S_{i-1}\times S_{k+1-i}\subset S_k$, i.e. combine together the summands corresponding to permutations of the first $i-1$ and of the last $k+1-i$ variables t_j's. Applying the identities (III) and (II) we get

$$\frac{(-1)^{i-1}}{(s-t_1)\dots(s-t_{i-1})} \times \frac{s^{k-i}}{(s-t_i)\dots(s-t_k)\cdot t_i\dots t_k}\ .$$

Now we take the sum over the cosets of the subgroup $S_{i-1}\times S_{k+1-i}\subset S_k$ and collect look like terms. Every denominator is the same, $(s-t_1)\dots(s-t_k)\cdot t_1\dots t_k=T(s)\cdot\tau_k$, whereas the numerators contain all possible products of $i-1$ of variables t_j's. $\square$

Remarks.

1. Let us consider $t_1,\dots,t_k$, s_1 as fixed numbers, and s, s_2 as variables. Then the left-hand side of every identity is nothing but a partial fraction decomposition of the function from the right-hand side. This interpretation, indicated by V. Lin, leads to another proof of the identities.

2. As A. Varchenko pointed out, our identity (III) for $s=0$ follows from the coincidence of the forms Ω^{sl_2} and $\widetilde{\Omega}^{sl_2}$ from [RStV, page 2]. Notice that for arbitrary s the identity (III) can be obtained from this particular case by substitution $\mathbf{t}\mapsto\mathbf{t}-s$. Similarly, the substitution $s\mapsto s_1-s_2$, $\mathbf{t}\mapsto\mathbf{t}-s_2$ transforms the identity (II) into the identity (I).

3. A remark of the referee is that the identity (III) could be deduced from the identity (I). Indeed if we consider (I) as a function of a complex variable s_1 and take the residues of both sides at infinity, then we get (III).

3. Bethe vectors

Here we recall the constructions for the tensor product of $n=2$ modules corresponding to points $z_0=0$ and $z_1=1$. For $n>2$ (and for any simple Lie algebra), see [FeFRe, ReV].

3.1. **Subspace of singular vectors in L.** Denote by $\{e_i,f_i,h_i\}_{i=1}^N$ the standard Chevalley generators of $sl_{N+1}(\mathbb{C})$,

$$[h_i,e_i]=2e_i\,,\ [h_i,f_i]=-2f_i\,,\ [e_i,f_i]=h_i\,;\qquad [h_i,h_j]=0\,,\ [e_i,f_j]=0\ \text{if}\ \ i\neq j\,.$$

Let $\mathfrak{h}$ be the Cartan subalgebra and $\mathfrak{h}^*$ its dual,

$$\mathfrak{h}^*\ =\ \mathbb{C}\{\lambda_1,\dots,\lambda_{N+1}\}\,/\,(\lambda_1+\dots+\lambda_{N+1}=0)\ ,$$

with the standard bilinear form $(\cdot,\cdot)$. The simple positive roots are $\alpha_i=\lambda_i-\lambda_{i+1}$, $1\le i\le N$,

$$(\alpha_i,\alpha_i)=2\,;\quad (\alpha_i,\alpha_j)=0\,,\ \text{if}\ |i-j|>1\,;\quad \text{and}\quad (\alpha_i,\alpha_j)=-1\,,\ \text{if}\ |i-j|=1\,.$$

Let $\Lambda(1)$ and $\Lambda(0)$ be integral dominant weights, and $\mathbf{k}=(k_1,\dots,k_N)$ be a vector with nonnegative integer coordinates such that

$$\Lambda(\mathbf{k}):=\Lambda(1)+\Lambda(0)-k_1\alpha_1-\dots-k_N\alpha_N$$

is an integral dominant weight as well. Denote by

$$\mathrm{Sing}_{\mathbf{k}}L:=\{\mathbf{v}\in L\,|\,h_i\mathbf{v}=(\Lambda(\mathbf{k}),\alpha_i)\mathbf{v},\ e_i\mathbf{v}=0,\ i=1,\dots,N\}$$

the subspace of singular vectors in $L=L_{\Lambda(1)}\otimes L_{\Lambda(0)}$ of weight $\Lambda(\mathbf{k})$.

3.2. **Bethe system associated with** $\mathrm{Sing}_{\mathbf{k}} L$**.** For every $i = 1, \dots, N$ introduce a set of k_i auxiliary variables associated with the root α_i,

$$t(i) := (\, t_1(i)\,,\ \dots\,,\ t_{k_i}(i))\ ,$$

and write $\mathbf{t} := (\, t(1)\,,\ \dots\,, t(N)\,)$.

The *Bethe system* is the following system of equations on variables $t_l(i)$,

$$\sum_{s\neq l} \frac{2}{t_l(i) - t_s(i)} - \sum_{s=1}^{k_{i-1}} \frac{1}{t_l(i) - t_s(i-1)} - \sum_{s=1}^{k_{i+1}} \frac{1}{t_l(i) - t_s(i+1)} - \frac{(\Lambda(0), \alpha_i)}{t_l(i)} - \frac{(\Lambda(1), \alpha_i)}{t_l(i) - 1} \;=\; 0\,,$$

here $1 \le i \le N,\ 1 \le l \le k_i$.

Every solution $\mathbf{t}^{(0)}$ to this system determines a *Bethe vector* $\mathbf{v}(\mathbf{t}^{(0)}) = \mathbf{v}_{\mathbf{k}}(\mathbf{t}^{(0)}) \in \mathrm{Sing}_{\mathbf{k}} L$. The function $\mathbf{v}(\mathbf{t})$ is described in Section 3.4.

3.3. **Bethe equations in terms of polynomials** $T_1(x), \dots, T_N(x)$**.** We use the notation introduced in Section 2.

Proposition 1. *Assume all the roots of* $T(x)$ *to be simple. Then*

$$\frac{T'(x)}{T(x)} = \sum_{j=1}^{k} \frac{1}{x - t_j}\,; \qquad \frac{T''(t_i)}{T'(t_i)} = \sum_{j\neq i} \frac{2}{t_i - t_j}\,.$$

Proof. The first equation is just the logarithmic derivative of T.
We have

$$T'(x) \;=\; \left(\sum_{j=1}^{k} \frac{1}{x - t_j} \right) \cdot T(x)\,.$$

Derivation of this equation gives

$$T''(x) \;=\; \left(\sum_{j=1}^{k} \frac{1}{x - t_j} \right)' \cdot T(x) + \left(\sum_{j=1}^{k} \frac{1}{x - t_j} \right)^2 \cdot T(x)\,.$$

Therefore

$$\frac{T''(x)}{T(x)} \;=\; -\sum_{j=1}^{k} \frac{1}{(x-t_j)^2} \;+\; \left(\sum_{j=1}^{k} \frac{1}{x - t_j} \right)^2 \;=\; 2 \sum_{1\le j<l\le k} \frac{1}{(x-t_j)(x-t_l)}\,.$$

We have

$$\frac{T''(x)}{T'(x)} \;=\; \frac{2 \displaystyle\sum_{1\le j<l\le k} \frac{1}{(x-t_j)(x-t_l)}}{\displaystyle\sum_{j=1}^{k} \frac{1}{x-t_j}} \;=\; \frac{2 \displaystyle\sum_{1\le j<l\le k} (x-t_1)\dots\widehat{(x-t_j)}\dots\widehat{(x-t_l)}\dots(x-t_k)}{\displaystyle\sum_{j=1}^{k} (x-t_1)\dots\widehat{(x-t_j)}\dots(x-t_k)}\,.$$

Substitution $x = t_i$ gives

$$\frac{T''(t_i)}{T'(t_i)} = \frac{2\sum\limits_{j\neq i}(t_i-t_1)\dots\widehat{(t_i-t_j)}\dots\widehat{(t_i-t_i)}\dots(t_i-t_k)}{(t_i-t_1)\dots\widehat{(t_i-t_i)}\dots(t_i-t_k)},$$

and the division finishes the proof. □

Now we can re-write the Bethe system in terms of polynomials $T_1(x),\dots,T_N(x)$, where

$$T_i(x) = (x-t_1(i))\ \dots\ (x-t_{k_i}(i))\ .$$

We have

$$\frac{T_i''(t_l(i))}{T_i'(t_l(i))} - \frac{T_{i-1}'(t_l(i))}{T_{i-1}(t_l(i))} - \frac{T_{i+1}'(t_l(i))}{T_{i+1}(t_l(i))} - \frac{(\Lambda(0),\alpha_i)}{t_l(i)} - \frac{(\Lambda(1),\alpha_i)}{t_l(i)-1} = 0\,,$$

for $1\le i\le N,\ 1\le l\le k_i$.

3.4. **Function $\mathbf{v}(\mathbf{t})$.** The function $\mathbf{v}(\mathbf{t})$ has been obtained in [SV, Sections 6,7] in general setting (see also [MV1], where it is called *the universal weight function*). Below we rewrite this function for the weight $\Lambda(\mathbf{k})$ in the tensor product of two modules $L = L_{\Lambda(1)}\otimes L_{\Lambda(0)}$ corresponding to points $z_1 = 1$ and $z_0 = 0$. Generic case can be found in [FeFRe, ReV].

To begin with, we construct the vectors that generate the subspace $L_{\Lambda(\mathbf{k})}\subset L$ of weight $\Lambda(\mathbf{k})$. In general, their number is greater than the dimension of that subspace, so they are linearly dependent.

Consider all pairs of words $(\mathbf{F}_1,\mathbf{F}_0)$ in letters $f_1,\dots,f_N$ subject to the condition that the total number of occurrences of letter f_i in both words is precisely k_i. Our vectors will be labeled by these pairs. Namely, we may think about words $\mathbf{F}_1$ and $\mathbf{F}_0$ as elements of the universal enveloping algebra of sl_{N+1} that naturally act on the spaces $L_{\Lambda(1)}$ and $L_{\Lambda(0)}$, respectively. Fix the highest weight vectors $\mathbf{v}_1\in L_{\Lambda(1)}\,,\ \mathbf{v}_0\in L_{\Lambda(0)}$. Then the vector

$$\mathbf{w}_{(\mathbf{F}_1,\mathbf{F}_0)} := \mathbf{F}_1\mathbf{v}_1\ \otimes\ \mathbf{F}_0\mathbf{v}_0$$

has weight $\Lambda(\mathbf{k})$, and all such vectors generate the weight space $L_{\Lambda(\mathbf{k})}$.

Now we define $\mathbf{v}_{\mathbf{k}}(\mathbf{t})$ as a linear combination

$$\mathbf{v}_{\mathbf{k}}(\mathbf{t})\ :=\ \sum_{(\mathbf{F}_1,\mathbf{F}_0)}\ \omega_{(\mathbf{F}_1,\mathbf{F}_0)}(\mathbf{t})\,\mathbf{w}_{(\mathbf{F}_1,\mathbf{F}_0)}\,,$$

where $\omega_{(\mathbf{F}_1,\mathbf{F}_0)}(\mathbf{t})$ are certain rational functions. We will construct these functions in two steps described below. Write

$$\mathbf{F}_1 = f_{i_1}\dots f_{i_{s_1}}\,,\qquad \mathbf{F}_0 = f_{j_1}\dots f_{j_{s_0}}\,,\qquad \mathbf{F}_1\,\mathbf{F}_0 = f_{i_1}\dots f_{i_{s_1}}\,f_{j_1}\dots f_{j_{s_0}}\,.$$

The length of the word $\mathbf{F}_1\mathbf{F}_0$ equals $s_1+s_0 = k_1+\dots+k_N$.

The first step is to translate $(\mathbf{F}_1,\mathbf{F}_0)$ into a rational function $g_{(\mathbf{F}_1,\mathbf{F}_0)}(\mathbf{t})$ of $\mathbf{t}$. For every $i = 1,\dots,N$, we replace the first occurrence (from left to right) of f_i in the word $\mathbf{F}_1\mathbf{F}_0$ by the variable $t_1(i)$; the second occurrence by the variable $t_2(i)$; and so on up to the last, k_i-th, occurrence, where f_i will be replaced by $t_{k_i}(i)$. We will get a pair of words in $\mathbf{t}$. We

augment these two words by 1 and 0, according to the values of z_1 and z_0, and thus get the row,

$$t_{a_1}(i_1)\, t_{a_2}(i_2)\ \ldots\ t_{a_{s_1}}(i_{s_1})\, 1\,, \qquad t_{b_1}(j_1)\, t_{b_2}(j_2)\ \ldots\ t_{b_{s_0}}(j_{s_0})\, 0\,,$$

in which every variable $t_l(i)$ from $\mathbf{t}$ appears precisely once. This row defines the fraction

$$g_{(\mathbf{F}_1,\mathbf{F}_0)}(\mathbf{t}) := \frac{1}{\big(t_{a_1}(i_1) - t_{a_2}(i_2)\big)\big(t_{a_2}(i_2) - t_{a_3}(i_3)\big)\cdots\big(t_{a_{s_1-1}}(i_{s_1}) - t_{a_{s_1}}(i_{s_1})\big)\big(t_{a_{s_1}}(i_{s_1}) - 1\big)}$$

$$\times\ \frac{1}{\big(t_{b_1}(j_1) - t_{b_2}(j_2)\big)\big(t_{b_2}(j_2) - t_{b_3}(j_3)\big)\cdots\big(t_{b_{s_0-1}}(j_{s_0}) - t_{b_{s_0}}(j_{s_0})\big)t_{b_{s_0}}(j_{s_0})}\,.$$

The second step is the symmetrization of $g_{(\mathbf{F}_1,\mathbf{F}_0)}(\mathbf{t})$. Let $S_{\mathbf{k}}$ denote the group of permutations of variables

$$\mathbf{t} = (t_1(1), \ldots, t_{k_1}(1),\ t_1(2), \ldots, t_{k_2}(2),\ \ldots\ ,\ t_1(N), \ldots, t_{k_N}(N))$$

that permute variables $t_1(i), \ldots, t_{k_i}(i)$ within their own, i-th, set, for every $i = 1, \ldots, N$. Thus $S_{\mathbf{k}}$ is isomorphic to the direct product $S_{k_1} \times S_{k_2} \times \cdots \times S_{k_N}$ of permutation groups.

For a function $g(\mathbf{t})$ define the symmetrization by the formula

$$\mathcal{S}ym_{\mathbf{k}}[g] := \sum_{\pi \in S_{\mathbf{k}}} g\Big(\pi\big(t_1(1)\big), \ldots, \pi\big(t_{k_N}(N)\big)\Big)\,.$$

Finally we set

$$\omega_{(\mathbf{F}_1,\mathbf{F}_0)}(\mathbf{t}) := \mathcal{S}ym_{\mathbf{k}}\left[g_{(\mathbf{F}_1,\mathbf{F}_0)}(\mathbf{t})\right]\,.$$

Notice that the universal weight function $\mathbf{v}_{\mathbf{k}}(\mathbf{t})$ is defined for any, not necessarily dominant, weight $\Lambda(\mathbf{k})$ presented in $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$. However in the Bethe Ansatz it is used only when $\Lambda(\mathbf{k})$ is the highest weight of an irreducible component of L.

4. Checking the non-triviality of Bethe vectors in examples

4.1. **Example $\Lambda(1) + \Lambda(0) - k\alpha_1$.** We assume $\Lambda(1)$ and $\Lambda(0)$ to be integral dominant weights and k be an integer such that

$$\Lambda(k,0) \ :=\ \Lambda(1) + \Lambda(0) - k\alpha_1$$

is the highest weight of an irreducible component of $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$. In this case Steinberg's formula implies that $f_1^k\,\mathbf{v}_0 \neq \mathbf{0}$ [Hu, Exercise 24.12]. We will show that the universal weight function $\mathbf{v}_k(\mathbf{t}) := \mathbf{v}_{(k,0,\ldots,0)}(\mathbf{t})$ never vanish. We have

$$\mathbf{k} = (k, 0, \ldots, 0)\,, \quad \mathbf{t} = (t(1))\ , \quad \mathbf{F}_0 = f_1^k\,, \quad g_{(\emptyset,\mathbf{F}_0)}(\mathbf{t}) = \frac{1}{(t_1 - t_2)\ldots(t_{k-1} - t_k)t_k}\,.$$

Simplifying the notation, write

$$t := (t_1, \ldots, t_k)\,, \qquad T(t) = \prod_{i=1}^{k}(x - t_i)\,, \qquad \omega_{k,0}(t) := \omega_{(\emptyset,\mathbf{F}_0)}(\mathbf{t})\,.$$

Theorem 2. *The projection of the vector $\mathbf{v}_k(\mathbf{t})$ to the subspace of $L_{\Lambda(k,0)}$ spanned by $\mathbf{v}_1 \otimes f_1^k\,\mathbf{v}_0$ is a non-zero vector.*

Proof. Notice that the domain of the function $\mathbf{v}_k(\mathbf{t})$ is given by the inequalities,

$$t_i \neq t_j\,, \quad t_i \neq 0\,, \quad 1 \leq i \neq j \leq k\,.$$

The considered projection of $\mathbf{v}_k(\mathbf{t})$ has the form $\omega_{k,0}(t)\mathbf{v}_1 \otimes f_1^k\,\mathbf{v}_0$, where

$$\omega_{k,0}(t) \;=\; \mathcal{S}ym_k\left[\frac{1}{(t_1-t_2)\dots(t_{k-1}-t_k)t_k}\right].$$

The identity (III) with $s=0$ gives

$$\omega_{k,0}(t) \;=\; \frac{(-1)^k}{T(0)} \;=\; \frac{1}{t_1\,t_2\dots t_k}\,,$$

and this fraction never vanishes. □

4.2. **Generalization to arbitrary** n. Theorem 2 has the following generalization to the universal weight function $\mathbf{v}(\mathbf{t})$ corresponding to the weight

$$\Lambda(k_1,\dots,k_m) = \sum_{i=1}^{n}\Lambda(i) \;-\; \sum_{i=1}^{m} k_i\alpha_i$$

in the tensor product

$$L = L_{\Lambda(1)} \otimes \dots \otimes L_{\Lambda(n)}$$

of n highest weight sl_{N+1}-representations marked by distinct complex numbers $z_1,\dots,z_n$.

Theorem 3. *Assume that $m \leq \min(n,N)$ and*

$$k_i \leq (\Lambda(i),\alpha_i)\,, \quad i=1,\dots,m\,.$$

Then the universal weight function $\mathbf{v}(\mathbf{t})$ corresponding to the weight $\Lambda(k_1,\dots,k_m)$ does not vanish.

Proof. Fix highest weight vectors $\mathbf{v}_i \in L_{\Lambda(i)}\,,\ \ i=1,\dots,n$. According to our assumptions, we have

$$f_i^{k_i}\mathbf{v}_i \neq \mathbf{0}\,, \quad i=1,\dots,m.$$

Consider the projection of $\mathbf{v}(\mathbf{t})$ to the one-dimensional subspace of L spanned by

$$f_1^{k_1}\mathbf{v}_1 \otimes \dots \otimes f_m^{k_m}\mathbf{v}_m \otimes \mathbf{v}_{m+1} \otimes \dots \otimes \mathbf{v}_n\,.$$

Applying the identity (III), one gets that the corresponding coefficient is equal to

$$\frac{(-1)^{k_1+\dots+k_m}}{T_1(z_1)\dots T_m(z_m)}\,,$$

where polynomials $T_i(x)$ are as in Section 3.3, and hence does not vanish. □

In particular, if the Bethe vector of the weight $\Lambda(k_1,\dots,k_m)$ exists, then it is a non-zero vector.

Returning to $n=2$, in the case when one of two modules is a symmetric power of the standard representation, we arrive at the following result.

Corollary 2. *If $L_{\Lambda(0)} = m\lambda_1$ and $\Lambda(k_1, k_2) = \Lambda(1) + \Lambda(0) - k_1\alpha_1 - k_2\alpha_2$ is the highest weight of an irreducible component of $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$, then the universal weight function $\mathbf{v}(\mathbf{t})$ corresponding to the weight $\Lambda(k_1, k_2)$ does not vanish. In particular, if $L_{\Lambda(1)}$ and $L_{\Lambda(0)}$ are sl_3-representations, then all Bethe vectors in L do not vanish.*

Proof. Elementary considerations with the Pieri formula [FuH, Proposition 15.25] show that the conditions

$$k_1 \le (\Lambda(0), \alpha_1)\,, \quad k_2 \le (\Lambda(1), \alpha_2)$$

are always fulfilled. In the sl_3 case all highest weights are clearly of the form $\Lambda(k_1, k_2)$. □

4.3. **Example $\Lambda(1) + \Lambda(0) - k\alpha_1 - \alpha_2 - \alpha_3$.** We assume that $\Lambda(0) = m\lambda_1$, $N \ge 3$, and $k \ge 1$ is an integer such that

$$\Lambda(k, 1, 1) := \Lambda(1) + \Lambda(0) - k\alpha_1 - \alpha_2 - \alpha_3$$

is the highest weight of an irreducible component of $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$. As before, the Pieri formula [FuH, Proposition 15.25] implies that $f_2 f_1^k \mathbf{v}_0 \ne \mathbf{0}$ and $f_3\mathbf{v}_1 \ne \mathbf{0}$ for any fixed highest weight vectors $\mathbf{v}_0 \in L_{\Lambda(0)}$ and $\mathbf{v}_1 \in L_{\Lambda(1)}$.

The module $L_{\Lambda(0)}$ is the m-th symmetric power of the standard sl_{N+1}-representation. Take $\mathbf{v}_0 = \epsilon_1^m$, where $\{\epsilon_i\}$ is a basis in the standard representation,

$$f_i\epsilon_i = \epsilon_{i+1}\,, \quad f_i\epsilon_j = \mathbf{0}\,, \quad i \ne j\,.$$

The subspace of weight $\Lambda(0) - k\alpha_1 - \alpha_2$ in $L_{\Lambda(0)}$ is one-dimensional and generated by the vector $\epsilon_1^{m-k}\,\epsilon_2^{k-1}\,\epsilon_3$.

There are three sets of auxiliary variables. We write

$$t(1) = (t_1, \dots, t_k)\,,\ t(2) = s\,,\ t(3) = r\,,\ \mathbf{t} = (t, s, r) = (t_1, \dots, t_k, s, r)\,,\ T(x) = \prod_{i=1}^{k}(x - t_i)\,.$$

The word $\mathbf{F}_0$ can be written as $\mathbf{F}_0 = f_1^{i-1} f_2 f_1^{k+1-i}$ for $i = 1, \dots, k+1$. Notice that $f_1^k f_2 \mathbf{v}_0 = \mathbf{0}$ for our choice of $\Lambda(0)$, therefore we assume that i varies from 1 to k and set

$$\omega_i(t, s, r) := \omega_{\left(f_3, f_1^{i-1} f_2 f_1^{k+1-i}\right)}(\mathbf{t})\,, \quad i = 1, \dots, k.$$

Theorem 4. *If $T'(s) \ne 0$, then the projection of the vector $\mathbf{v}_{k,1,1}(t, s, r)$ to the subspace $L_{\Lambda(k,1,1)}$ spanned by*

$$f_3\mathbf{v}_1 \otimes \left(\epsilon_1^{m-k}\,\epsilon_2^{k-1}\,\epsilon_3\right)$$

is a non-zero vector.

Proof. Notice that the domain of the function $\mathbf{v}_{k,1,1}(t, s, r)$ is given by the inequalities,

$$t_i \ne t_j\,, \quad , t_i \ne s\,, \quad , t_i \ne r\,, \quad , r \ne s\,, \quad t_i, s, r \ne 0, 1\,, \quad 1 \le i \ne j \le k.$$

The projection of vector $\mathbf{v}_{k,1,1}(t, s, r)$ to the chosen subspace has the form

$$\sum_{i=1}^{k} \omega_i(t, s, r)\mathbf{w}_i\,,$$

where $\mathbf{w}_i = f_3\mathbf{v}_1 \otimes f_1^{i-1} f_2 \, f_1^{k+1-i}\mathbf{v}_0$. Here f_2 stands at the i-th place from the left, and

$$\omega_i(t,s,r) = \frac{1}{r-1}\mathcal{S}ym_k\left[\frac{1}{(t_1-t_2)\dots(t_{i-2}-t_{i-1})(t_{i-1}-s)(s-t_i)(t_i-t_{i+1})\dots(t_{k-1}-t_k)t_k}\right].$$

An easy calculation shows that

$$\mathbf{w}_i \;=\; (k+1-i)\cdot m(m-1)\dots(m+1-k)\cdot f_3\mathbf{v}_1 \otimes \left(\epsilon_1^{m-k}\,\epsilon_2^{k-1}\,\epsilon_3\right)\;,$$

therefore the projection is

$$\left(\sum_{i=1}^{k}(k+1-i)\cdot\omega_i\,(t,s,r)\right)\cdot m(m-1)\dots(m+1-k)\cdot f_3\mathbf{v}_1 \otimes \left(\epsilon_1^{m-k}\,\epsilon_2^{k-1}\,\epsilon_3\right)\;.$$

It is convenient to use the notation introduced at the end of Section 2. The identity (IV) of Corollary 1 gives

$$\omega_i(t,s,r) = \frac{(-1)^{i-1}\;s^{k-i}\;\tau_{i-1}}{(r-1)T(s)\;\tau_k}\,,\quad 1\le i\le k\,.$$

Therefore

$$\sum_{i=1}^{k}(k+1-i)\cdot\omega_i\,(t,s,r)\;=\;\frac{T'(s)}{(r-1)T(s)\tau_k}\,,$$

and the statement of the theorem follows. □

Corollary 3. *If the Bethe vector $\mathbf{v}_{k,1,1}$ exists, then it does not vanish.*

Proof. We show that $T'(s)$ can not vanish at a solution of the Bethe system. The Bethe equation corresponding to the variable s is as follows,

$$\frac{1}{s-r}+\frac{T'(s)}{T(s)}+\frac{(\Lambda(1),\alpha_2)}{s-1}=0\,,$$

whereas the one corresponding to r has the form

$$\frac{1}{r-s}+\frac{(\Lambda(1),\alpha_3)}{r-1}=0\,,$$

see Section 3.3. Denote $(\Lambda(1),\alpha_2)=A$ and $(\Lambda(1),\alpha_3)=B$. Assuming $T'(s)=0$ one gets the following linear system with respect to s and r,

$$-Ar+(A+1)s=1\,,\quad (B+1)r-Bs=1\,.$$

The solution to this system is $r=s=1$ and contradicts to the conditions $r\neq s\neq 1$. □

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Sergei Chmutov
The Ohio State University, Mansfield
1680 University Drive
Mansfield, OH 44906
USA
chmutov@math.ohio-state.edu

Inna Scherbak
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, Tel Aviv 69978
Israel
scherbak@post.tau.ac.il